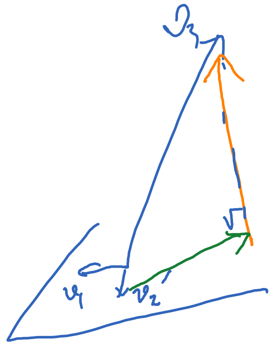
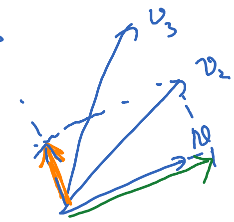
Lecture 27

Tuesday, December 8, 2020 3:59 PM

Goal: turn a basis into an orthogonal basis



$B = \{v_1, v_2, v_3\}$ basis of $\mathbb{R}^3$



$v_1 \longrightarrow v_1$

$v_2 \longrightarrow v_2 - \text{proj}_{v_1} v_2$ $\qquad v_2 = \text{proj}_{v_1} v_2 + \text{perp}_{v_1} v_2$

$v_2' = v_2 - \dfrac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1$ $\qquad v_3 =$

$v_3 \longrightarrow v_3 - \text{proj}_{\{v_1, v_2'\}} v_3 = v_3 - \text{proj}_{v_1} v_3 - \text{proj}_{v_2'} v_3$

$\mathbb{R}^3$

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$B = \{v_1, v_2, v_3\}$ $\qquad x_1, x_2, x_3 \qquad v_1, v_2, v_3$

$\qquad v_1, v_2, v_3 \qquad v_1', v_2', v_3'$

$v_1 \longrightarrow v_1$

$v_2 \longrightarrow v_2$

$v_3 \longrightarrow v_3 - \text{proj}_{v_1} v_3 - \text{proj}_{v_2} v_3$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{v_1 \cdot v_3}{v_1 \cdot v_1} v_1 - \frac{v_2 \cdot v_3}{v_2 \cdot v_2} v_2$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{5}{5} v_1 - 0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$\mathbb{R}^4$

$$V = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 0 \\ -1 \end{bmatrix} \right\} \qquad \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 3 & 0 & -1 \end{bmatrix} \xrightarrow{R_2 = R_2 + 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 7 & 6 & 7 \end{bmatrix}$$

$$\xrightarrow{R_2 = \frac{R_2}{7}} \quad R_1 = R_1 - 2R_2 \qquad \begin{bmatrix} 1 & 0 & \frac{9}{7} & 2 \\ 0 & 1 & \frac{6}{7} & 1 \end{bmatrix}$$

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

v_1, v_2, v_3, v_4

$v_1, v_2,$

$$v_3 \longrightarrow v_3 - \frac{v_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{v_3 \cdot v_1}{v_2 \cdot v_2}$$

$$v_4 \longrightarrow v_4 - \frac{v_4 \cdot v_1}{v_1 \cdot v_1} v_1 - \ldots$$

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* Special matrices

- Orthogonal matrix: $A^{-1} = A^T$
- Symmetric matrix: $A^T = A$

Orthogonal matrix : A

$$A^{-1} = A^T \rightsquigarrow AA^T = I_n \qquad A^TA = I_n$$

$$\begin{bmatrix} - & R_1 & - \\ - & R_2 & - \\ & \vdots & \\ - & R_n & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ R_1 & R_2 & \cdots & R_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

$R_1 \cdot R_1 = 1$ $\qquad$ $R_i \cdot R_i = 1 \rightsquigarrow \|R_i\| = 1$

$R_1 \cdot R_2 = 0$ $\qquad$ $R_i \cdot R_j = 0$

rows of an orthogonal matrix form an orthonormal basis of $\mathbb{R}^n$.

* Symmetric matrix

$$A = \begin{bmatrix} 1 & 5 & 6 \\ 5 & 2 & 7 \\ 6 & 7 & 3 \end{bmatrix} = A^T$$

diagonalizable

$$A = \underset{\sim}{P} D P^{-1}$$

can be picked so that it is orthogonal matrix